Dynamics and interaction of solitons on an integrable inhomogeneous lattice

V. V. Konotop
Universidade de Madeira, Colégio dos Jesuitas, Praça do Municipio, P 9000 Funchal, Portugal,
and Departamento de Física Teórica I, Facultad de Ciencias Físicas, Universidad Complutense, E-28040 Madrid, Spain

O. A. Chubykalo and L. Vázquez
Departamento de Física Teórica I, Facultad de Ciencias Físicas Universidad Complutense, E-28040 Madrid, Spain

(Received 18 February 1993)

We discuss different aspects of one-soliton and multisoliton dynamics governed by the lattice equation $i\dot{\psi}_n+(\psi_{n-1}+\psi_{n+1})(1+|\psi_n|^2)-2\gamma(t)n\psi_n=0$. We propose reduction of this equation, drastically simplifying its treatment by means of the inverse scattering technique.

PACS number(s): 03.20.+i, 03.40.Kf, 05.50.+q, 02.90.+p

I. INTRODUCTION

The present paper is devoted to the lattice equation

$$i\dot{\psi}_n+(\psi_{n+1}+\psi_{n-1})(1+|\psi_n|^2)-2\gamma(t)n\psi_n=0 , \quad (1)$$

in which $\gamma(t)$ is an arbitrary function of time and the dot hereafter stands for the time derivative. If $\gamma(t)$ is identically equal to zero, this equation becomes the well-known Ablowitz-Ladik model [1], which attracts much attention due to both its exact integrability and reduction to the nonlinear Schrödinger equation in the continuum limit. For $\gamma(t)\neq 0$ the last term in Eq. (1) is an on-site external potential linear in space and varying in time. A particular case $\gamma(t)\equiv\gamma=\text{const}$ has recently been studied by Scharf and Bishop [2]. The authors obtained a UV pair of Eq. (1) and, thus, stated that the model is integrable by means of the inverse scattering technique. Its one-soliton solution has been presented. On the basis of the explicit form of a one-soliton solution and of the detailed numerical study provided in Ref. [2], very interesting behavior of solitons in the model (1) with $\gamma(t)\equiv\text{const}$ has been observed. It turns out, in particular, that a soliton can be "trapped" by the linear external potential in the sense that its motion becomes periodic and a mean velocity of the forward motion is equal to zero. Respectively, in a case of two solitons, their interaction occurs periodically, while Ablowitz-Ladik solitons move in accordance with classical rules for solitons.

The model of Scharf and Bishop has an integrable and well-studied continuum limit [3,4], which is a nonlinear Schrödinger equation under linear potential. As it has been stated by Balakrishnan [4], a generalization of this equation corresponding to time-dependent amplitudes of a linear potential can be included as well into the scheme of the inverse scattering technique. One soliton solution of such a generalization has been found by Besieris [5]. Noting that the last evolution system is nothing but the continuum limit of Eq. (1), it has been pointed out in [6] that Eq. (1) can also be treated by the inverse scattering method. The zero-curvature condition is found by a simple generalization of the UV pair obtained in Ref. [2]. The explicit form of the one-soliton solution of Eq. (1) is obtained in Ref. [6].

The purpose of the present study is to outline some specific points of the inverse scattering scheme associated with Eq. (1), to represent the multisoliton solution, to give mathematical stipulations for some effects observed earlier, and to give a panoramic of soliton dynamics governed by Eq. (1).

Before getting into details we should list some physical applications of the model (1). In a general sense, Eq. (1) describes the evolution of solitons on a lattice affected by a linear potential with an amplitude varying in time. As was shown in Ref. [6] in the case of random function $\gamma(t)$, Eq. (1) can serve as an effective equation for dynamics of more general nonlinear random lattices in which $\gamma(t)n$ is replaced by a time δ-function-correlated process $\gamma_n(t)$. One more application of the system under discussion is related to soliton motion in a smooth potential. The last problem for the one-soliton dynamics has been studied in [2] in the framework of the collective coordinate approach. Now we show that Eq. (1) provides another approach to the problem. Indeed, let a soliton be localized on the distance of β^{-1} order (it will be clear from the consideration below that these arguments make sense also for some localized multisoliton pulses, to which application of the collective coordinate approach is problematic) and v is a soliton velocity. Also let l be a characteristic scale of the slowly varying external potential $W(n)$. Then under supposition $\beta l\gg 1$, the potential multiplied by ψ_n can be expanded into the Taylor series around $n=vt$:

$$W(n)\psi_n\approx\left[W(vt)-vt\left.\frac{\partial W(n)}{\partial n}\right|_{n=vt}\right]\psi_n+\left.\frac{\partial W(n)}{\partial n}\right|_{n=vt}n\psi_n .$$

Thus in this approximation after evident phase transformation the Ablowitz-Ladik model perturbed by the smooth potential $W(n)$ is reduced to Eq. (1).

The organization of the paper is as follows. In Sec. II we represent transformation of Eq. (1) allowing one to obtain easily multisoliton solutions with the help of the knowledge of the Ablowitz-Ladik model. Then, in Sec.

1063-651X/93/48(1)/563(6)/$06.00

III, we discuss features of one-soliton dynamics. Section IV is devoted to interaction of solitons. The results are summarized in Sec. V.

II. GENERAL SOLUTION

Using the UV pair of Ref. [2] it is not difficult to obtain the zero-curvature condition for Eq. (1). However, in doing so we come to the direct spectral problem having a time-dependent spectral parameter (unlike the unperturbed Ablowitz-Ladik model, it will move around the unit circle in the complex plane). Although because of Refs. [3,4] it is known how to proceed in this case, we prefer another way of dealing with the conventional spectral problem of the Ablowitz-Ladik model. There are two facts prompting us how to find this way. The first one is the rotation of the eigenvalues, which has been mentioned above and has to be prevented. Another prompt is the transformation, proposed by Tappert [7], for the continuous nonlinear Schrödinger equation affected by the linear potential. Therefore, considering an initial value problem, let us introduce a function

$$q_n(t)=e^{i(2n+1)\Gamma(t)}\psi_n(t) , \tag{2}$$

where

$$\Gamma(t)=\int_0^t dt\, \gamma(t) \tag{3}$$

is a real function. An equation for $q_n(t)$ follows immediately from (1),

$$i\dot{q}_n+(1+|q_n|^2)(q_{n-1}e^{2i\Gamma}+q_{n+1}e^{-2i\Gamma})+\gamma(t)q_n=0 . \tag{4}$$

This lattice model is exactly integrable. The zero-curvature condition

$$\dot{U}_n+U_nV_n-V_{n+1}U_n=0 , \tag{5}$$

for it is given by matrices

$$U_n=\begin{bmatrix} z & iq_n^* \\ iq_n & z^{-1}\end{bmatrix} , \tag{6}$$

$$V_n=i\begin{bmatrix} -z^2e^{2i\Gamma}-q_n^*q_{n-1}e^{2i\Gamma}-\dfrac{\gamma(t)}{2} & i\left[\dfrac{e^{-2i\Gamma}}{z}q_{n-1}^*-e^{2i\Gamma}zq_n^*\right] \\ i\left[\dfrac{e^{-2i\Gamma}}{z}q_n-e^{2i\Gamma}zq_{n-1}\right] & z^{-2}e^{-2i\Gamma}+q_nq_{n-1}^*e^{-2i\Gamma}+\dfrac{\gamma(t)}{2}\end{bmatrix} \tag{7}$$

(hereafter the asterisk stands for complex conjugation). Since only the V_n matrix differs from that of the Ablowitz-Ladik model, we have now to specify only the time dependence of the scattering data. The method to obtain it is well known [1]. Omitting some details we represent the result

$$T(z;t)=\exp\left[-i\frac{\Omega(t)+\Gamma(t)}{2}\sigma_3\right]T(z,0)\times\exp\left[i\frac{\Omega(t)+\Gamma(t)}{2}\sigma_3\right] \tag{8}$$

and

$$\bar{C}_k(t)=\bar{C}_{k,0}e^{-i[\Omega_k(t)+\Gamma(t)]} . \tag{9}$$

Here

$$T(z)=\begin{bmatrix} a(z) & -b^*(z) \\ b(z) & a^*(z)\end{bmatrix} \tag{10}$$

is a transfer matrix associated with the eigenvalue problem (6),

$$\Omega(t)=\int_0^t dt\, \omega(z), \quad \Omega_k(t)=\int_0^t dt\, \omega(z_k) , \tag{11}$$

$$\omega(z)=z^2e^{2i\Gamma}+z^{-2}e^{-2i\Gamma} , \tag{12}$$

σ_3 is Pauli matrix, and other designations are from Ref. [1].

Now, using the results of investigation of the inverse problem, obtained in Ref. [1], we can represent a multisoliton solution of Eq. (1) in the conventional form

$$\psi_n=ie^{-i(2n+1)\Gamma(t)}\frac{\Delta_{kj}^{(1)}}{\Delta_{kj}} , \tag{13}$$

where

$$\Delta_{kj}=\delta_{kj}+4\sum_{l=1}^{N}\bar{C}_k^*(t)\bar{C}_l(t)\alpha_{kl}\alpha_{jl} , \tag{14}$$

$$\Delta_{kj}^{(1)}=\begin{vmatrix} & 2\bar{C}_1^*(t)(\bar{z}_1^*)^{n-1} \\ \Delta_{kj} & \vdots \\ & 2\bar{C}_N^*(t)(\bar{z}_N^*)^{n-1} \\ (\bar{z}_1^*)^{n+1}\cdots(\bar{z}_N^*)^{n+1} & 0\end{vmatrix} , \tag{15}$$

$$\alpha_{ij}=(\bar{z}_i^*\bar{z}_j)^n\tilde{\alpha}_{ij}, \quad \tilde{\alpha}_{ij}=\frac{\bar{z}_i^*\bar{z}_j}{1-(\bar{z}_i^*\bar{z}_j)^2} , \tag{16}$$

and the time dependence of $\bar{C}_k(t)$ from Eq. (9) is taken into account.

In a particular case of a single eigenvalue of the operator U_n inside the unit circle, $\bar{z}_1=\exp(-w+i\theta)$, $w>0$, we obtain one-soliton solution in the form [6]

$$\psi_n = \frac{\sinh(2w)\exp\left\{\dfrac{i\theta_0}{2}+\dfrac{i\phi(t)}{2}-2in[\Gamma(t)+\theta]\right\}}{\cosh[2nw-X(t)-X_0]} , \qquad (17)$$

where functions $\phi(t)$ and $X(t)$ are defined by integrals

$$\phi(t)=4\cosh(2w)\int_0^t \cos\{2[\Gamma(t_0)+\theta]\}dt_0 , \qquad (18)$$

$$X(t)=2\sinh(2w)\int_0^t \sin\{2[\Gamma(t_0)+\theta]\}dt_0 , \qquad (19)$$

and

$$\theta_0=-\arg\bar{C}_{1,0}+\frac{\pi}{2}, \quad X_0=-\ln\left[\frac{|\bar{C}_{1,0}|}{\sinh(2w)}\right] . \qquad (20)$$

III. SIMPLE DYNAMICS OF A SINGLE SOLITON

Two particular cases of $\gamma(t)\equiv$const and $\gamma(t)$ being a white noise have been considered in Refs. [2,6]. In both cases the behavior of a discrete soliton essentially differs from its continuous analogue. Generally speaking, the evolution of a single soliton depends on the function $\Gamma(t)$, which is a constant for the Ablowitz-Ladik model and is proportional to t in the case of a constant linear potential. Now we show that there exists a representation giving a quite simple picture of one-soliton dynamics in a generic case.

The Ablowitz-Ladik lattice, as an integrable system, possesses an infinite number of conservation laws. The first ones (usually they are combined into one complex integral) can be written as follows (see, e.g., [1]):

$$P=i\sum(\psi_n\psi_{n-1}^*-\psi_n^*\psi_{n-1}) , \qquad (21)$$

$$E=\sum(\psi_n\psi_{n-1}^*+\psi_n^*\psi_{n-1}) . \qquad (22)$$

They can be treated as a momentum and an energy of the pulse. In our case these quantities become dependent on time. An infinite number of integrals exist for the problem we are dealing with and can be restored with the help of integrals of the Ablowitz-Ladik model [1,8]. It follows from the fact that these quantities are determined by the associated spectral problem. The linear spectral problem associated with the equation for $q_n(t)$ coincides with that of the Ablowitz-Ladik model which gives the conservative quantities of Eq. (4). In order to write down the integrals of Eq. (1) in terms of $\psi_n(t)$ we should use the ansatz (2). However, we concentrate now on functions $E(t)$ and $P(t)$. Using the above facts one immediately arrives at the "dispersion relation"

$$E^2+P^2=C^2 , \qquad (23)$$

where C is a positive constant defined by initial conditions. In the one-soliton case $C=4\sinh(2w)$ (see [2] for the idea of calculations).

Thus in the phase space (E,P) the trajectory of a soliton is a circle of a constant radius C. This allows us holding the sense of E and P to find a curious qualitative analogy between a lattice soliton and a particle in a periodic potential.

To this end we recall well-known facts of the solid-state physics [9]. In the absence of an external electric field, an electron in a periodic potential moves uniformly. Due to a constant electric field (or linear potential in our terms) a quasimomentum of a particle changes and moves towards the boundary of the Brillouin zone (in the quasimomentum space). Being unable to cross the boundary, a particle undergoes Bragg reflection. Thus oscillations of the electron in the quasimomentum space appear and result in oscillations in a usual space (this effect is called Bloch oscillations).

We have similar effect in our case. Indeed, in the absence of perturbation a soliton moves evenly. The external field $\gamma(t)$ results in a changing of the momentum P, which now evolves in accordance with the law

$$\dot{P}=-2\gamma(t)E, \quad \dot{E}=2\gamma(t)P \qquad (24)$$

[this formula is obtained directly from (21) and (22) and is valid for any pulse rather than only for a single soliton]. If $\gamma=$const, then P increases (or decreases) with the time. As follows from (23) the absolute value of the pulse momentum cannot be larger than C (in our treatment this quantity plays a part of the boundary of "Brillouin zone," in which energy E is equal to zero). Hence, after the absolute value of $P(t)$ having reached $4\sinh(2w)$, it decreases. Thus the "Bloch oscillations" of a soliton appear [10]. It is these oscillations that have been observed under linear potential in a number of numerical experiments of Ref. [2] (see Figs. 1 and 5 there). Comparing the outcomes with those in the continuum limit we come to the conclusion that the discreteness affects a soliton like a periodic field affects a particle.

Another manifestation of the oscillations under discussion is a "localization" by the linear potential (under this term we refer to a localized perturbation of the lattice which does not move in space). Indeed, let the parameters of a soliton satisfy the requirement

$$\frac{\sinh(2w)}{2\gamma}[\sin(2\theta)+\cos(2\theta)]\leq 1 , \qquad (25)$$

which means in fact that the soliton width is greater than the amplitude of the oscillations. Then, the solution of Eq. (1) will look like a localized in space excitation (see Fig. 1), which has zero velocity of the forward motion.

In a generic case the system (24) is trivially solved giving

$$\begin{aligned} P(t)&=P_0\cos[\Gamma(t)]-E_0\sin[\Gamma(t)] , \\ E(t)&=P_0\sin[\Gamma(t)]+E_0\cos[\Gamma(t)] , \end{aligned} \qquad (26)$$

where P_0 and E_0 are initial values of the momentum and the energy, and $\Gamma(t)$ is defined by (3).

As it follows from Eqs. (17)–(19) the velocity of the soliton is determined by the behavior of the functions

$$\begin{Bmatrix} f_c(t) \\ f_s(t) \end{Bmatrix} = \int_0^t d\tau \begin{Bmatrix} \cos \\ \sin \end{Bmatrix} [2\Gamma(\tau)] . \qquad (27)$$

Now we briefly describe the cases of periodic and random $\gamma(t)$.

If the amplitude of the linear potential is a periodic function, say

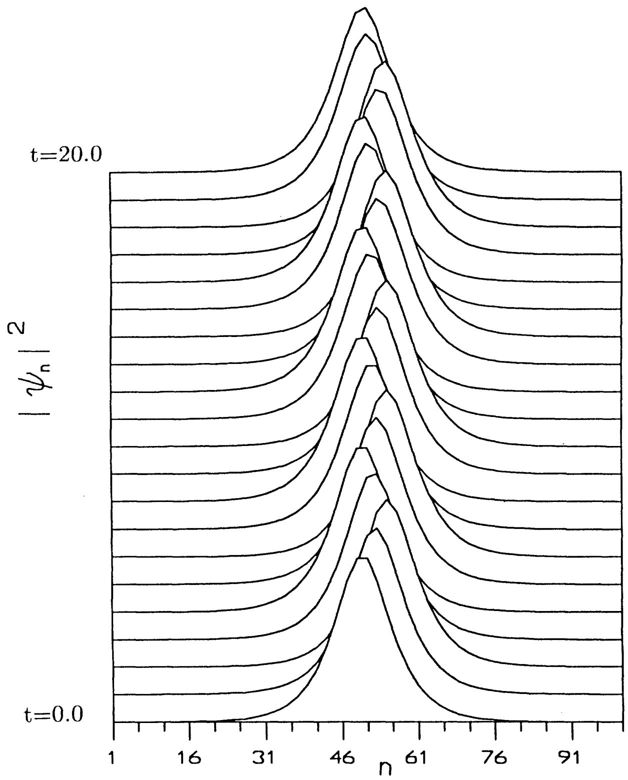

FIG. 1. Dynamics of a single soliton localized by linear potential with $\gamma=0.8$ ($w=0.1$, $\theta=0.0$, and $X_0=10.0$).

$$\gamma(t)=\sin(\omega t) , \tag{28}$$

the oscillations are accompanied by the motion in the forward direction (again analogously to a particle in a periodic potential). Recalling the one-soliton solution (17) we find a temporary average velocity of the forward motion

$$v_{\rm av}=\frac{\omega}{4\pi w}\left[X\left[t+\frac{2\pi}{\omega}\right]-X(t)\right] =\frac{\sinh(2w)}{w}\sin\left[2\theta+\frac{2}{\omega}\right]\mathcal{J}_0\left[\frac{2}{\omega}\right] , \tag{29}$$

where $\mathcal{J}_0(\ )$ is the Bessel function. Therefore, the direction of motion depends on the frequency ω of the external force. Note that there exists a set of frequencies ω at which $v_{\rm av}=0$.

The representation (26) allows one to make some general statements about soliton dynamics in the case of random $\gamma(t)$. In particular, if $\gamma(t)$ is a Gaussian process, $\Gamma(t)$ has a normal distribution, and one immediately calculates by direct averaging

$$\langle P\rangle=P_0e^{-\sigma^2(t)},\quad \langle E\rangle=E_0e^{-\sigma^2(t)} , \tag{30}$$

where

$$\sigma^2(t)=2\int_0^t d\tau_1\int_0^t d\tau_2\langle\gamma(\tau_1)\gamma(\tau_2)\rangle , \tag{31}$$

and angular brackets stand for averaging over all realizations of $\gamma(t)$. Asymptotically, when $T\gg t_c$, [t_c being a correlation radius of $\gamma(t)$ in time], $\sigma^2(t)$ grows and, consequently, averaged quantities tend to zero [unlike the case of periodic $\gamma(t)$]. Meantime, as it is easy to show, both $\langle P^2\rangle$ and $\langle E^2\rangle$ go to $(P_0^2+E_0^2)/2$. One can formulate a more general statement about statistics of E and P: all odd momenta tend to zero with time, while the absolute value of each even momentum of the order of $2k$ (k being integer) goes to $2^{-k}(P_0^2+E_0^2)^k$ (note that the result is valid for any pulse rather than for a single soliton only). This means that the distribution of E and P goes to a stationary one.

IV. ON MULTISOLITON SOLUTION

Passing to the discussion of the multisoliton dynamics we should point out that the main information has already been obtained [see Eqs. (13)–(16), (26), and some results of the preceding section]. Nevertheless, now we want to discuss in more details some qualitative features of multisoliton pulses of Eq. (1).

Recalling the numerical results of Ref. [2] on two soliton interactions we can state that the period of the processes observed there is to be nothing else but the period of the functions $f_{c,s}$, i.e., π/γ. Moreover, in the constant external field, any motion will be periodic [it follows from Eqs. (11) and (12)].

Then it may be found that the localized pulses may also be multisoliton pulses. As an example, we will consider a two-soliton solution, which can be represented in the following form:

$$\psi_n(t)=\frac{2i}{\Delta}e^{-2n\Gamma(t)}\{\nu_1e^{X_1/2-2nw_1}+\nu_2e^{X_2/2-2nw_2} +\mu_{12}e^{X_1+X_2/2-2n(2w_1+w_2)} +\mu_{21}e^{X_2+X_1/2-2n(2w_2+w_1)}\} , \tag{32}$$

where

$$\nu_i=\bar{C}^*_{i,0}e^{-2i\theta_in+i(\phi_i/2)} , \tag{33}$$

$$\mu_{ij}=4|\bar{C}_{i,0}|^2\bar{C}^*_{j,0}e^{-2i\theta_jn+i(\phi_j/2)} \times\left[\tilde{\alpha}^2_{ji}+\tilde{\alpha}^2_{ii}-\tilde{\alpha}_{ii}\tilde{\alpha}_{ji}\left[\frac{\bar{z}^{*2}_1+\bar{z}^{*2}_2}{\bar{z}^*_1\bar{z}^*_2}\right]\right] , \tag{34}$$

$$\Delta=1+\zeta e^{X_1+X_2-4(w_1+w_2)n} +\sum_{i,j}\xi_{ij}e^{(X_i+X_j)/2-2(w_i+w_j)n} , \tag{35}$$

$$\xi_{ij}=4\bar{C}^*_{i,0}\bar{C}_{j,0}\tilde{\alpha}^2_{ij}e^{i(\phi_i-\phi_j)/2-2i(\theta_i-\theta_j)n} , \tag{36}$$

$$\zeta=16|\bar{C}_{1,0}|^2|\bar{C}_{2,0}|^2[\tilde{\alpha}_{11}\tilde{\alpha}_{22}-\tilde{\alpha}_{12}\tilde{\alpha}_{21}]^2 , \tag{37}$$

and the parameters X_i and ϕ_i are determined by Eqs. (18) and (19) with respective subindices, and the free parameters w_k and θ_k are introduced as $\bar{z}_k=\exp(-w_k+i\theta_k)$. One can estimate the region of parameters giving localization. Figure 2 shows two evident pictures of different qualitative behavior of the two-soliton solution in the constant on-site potential (in addition to that demonstrated in Ref. [2]). In Fig. 2(a) one can find localized (and breathing) two-soliton solution. It is seen that we do not have two single pulses, and in this sense the name "two solitons" becomes more formal, designating only the presence of two discrete eigenvalues of the associated linear problem. Note that this situation differs from the case of the two solitons of the Ablowitz-Ladik model having equal velocities: even finite change of the parameters $\bar{z}_k$ and $\bar{C}_{k,0}$ does not change the qualitative behavior of the soliton. Figure 2(b) explicitly shows the opposite case, when the two-soliton solution consists of "interacting" solitons. As it is evident, similar pictures can be found for any multisoliton solution.

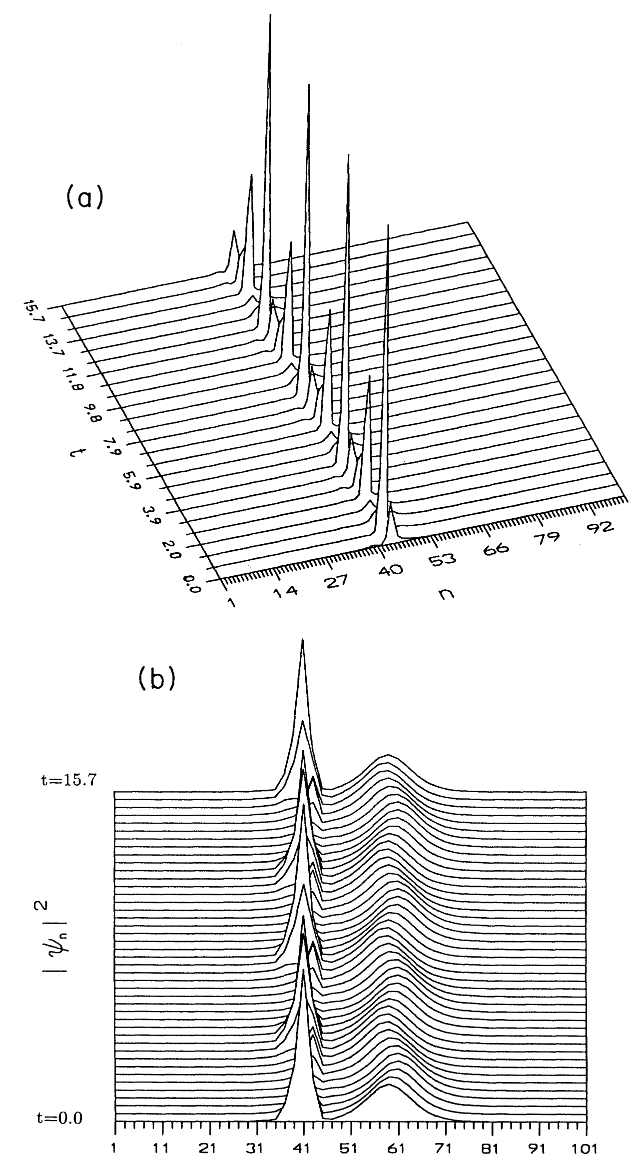

FIG. 2. Dynamics of two soliton solution localized by the linear potential with $\gamma=0.8$, $\theta_1=0.0$, $\theta_2=-0.5$, $\bar{C}_1=0.01$, and $\bar{C}_2=10.0$. (a) Breathing two-soliton solution ($w_1=0.4$, $w_2=-0.1$), (b) two "interacting" solitons ($w_1=0.4, w_2=0.08$).

As we have found, the force periodic in time results in a forward motion of solitons. Hence we can study how the changes in the asymptotic superposition law or principle occur due to the external field. Direct analysis of (32) yields that, in the case of two solitons, the second soliton at $n\to\pm\infty$ is described by

$$\psi_n=\frac{\sinh(2w_2)\exp\left\{i\left[\frac{\phi_2}{2}+\frac{\theta_{02}}{2}-2n\,[\theta_2+\Gamma(t)]+\chi_n^{(2)}\right]\right\}}{\cosh[2nw_2-X_2(t)-X_{02}-\delta_n^{(2)}]}\,, \tag{38}$$

where

$$\chi_n^{(2)}\to\begin{cases}0 \quad \text{at } n\to-\infty\\ 2[2\theta_1-\arg\{(\bar{z}_2^2-\bar{z}_1^2)(1-\bar{z}_1^2\bar{z}_2^{*2})\}] \\ \qquad\qquad\qquad \text{at } n\to+\infty\,,\end{cases} \tag{39}$$

$$\delta_n^{(2)}\to\begin{cases}0 \quad \text{at } n\to-\infty\\ \ln\dfrac{(\bar{z}_1^2-\bar{z}_2^2)(\bar{z}_1^{*2}-\bar{z}_2^{*2})}{[1-(\bar{z}_1\bar{z}_2^*)^2][1-(\bar{z}_2\bar{z}_1^*)^2]} \quad \text{at } n\to+\infty\,,\end{cases} \tag{40}$$

and ϕ_2, X_2, θ_{02}, and X_{02} are defined according to Eqs. (18)–(20). The interaction of solitons in this case leads both to the standard phase shift (40) and to the appearance of the additional phase χ_n^2. However, both these quantities do not depend explicitly on $\gamma(t)$. The only requirements necessary for (38)–(40) to be valid are $X_i(t)\to\pm\infty$ at $t\to\pm\infty$.

V. CONCLUDING REMARKS

To conclude, we have considered a generalized version of the Ablowitz-Ladik lattice, which includes a coefficient proportional to a site number and having an arbitrary dependence on time. The model being exactly integrable turns out to be closely related to the original Ablowitz-Ladik one from the mathematical viewpoint. Meanwhile, the dynamics of the solitons show a number of basic differences. One of them is the periodic motion of solitons under constant linear potential. These differences are stipulated by the discreteness and disappear when the corresponding continuous versions are considered [3].

Speaking about the transition to the continuous limit, it is appropriate to note that, in fact, the change of variables corresponding to the ansatz (2) has the form

$$\begin{aligned}\psi(t,x)&=\exp\left[2ix\,\Gamma(t)-4i\int dt'\,\Gamma^2(t')\right]q(t;\xi)\,,\\ \xi&=x+4\int\Gamma(t)dt\,.\end{aligned} \tag{41}$$

Here $\psi(t,x)$ and $q(t,\xi)$ have to be treated as respective continuous limits of $\psi_n(t)$ and $q_n(t)$. As is evident, this reduction is nothing but a generalization of Tappert's insertion [7] [in contrast to the discrete case, now we have an exact transformation of the "perturbed" equation for $\psi(t,z)$ to the nonlinear Schrödinger equation for the function $q(t,\xi)$ in terms of (t,ξ)].

ACKNOWLEDGMENTS

V.V.K. acknowledges the financial support of Complutense University of Madrid through a sabbatical program. O.A.C. is grateful to the Ministery of Education and Science of Spain for support. This work has been partially supported by the Comunidad de Madrid under Grant No. C174/91.

[1] M. Ablowitz and J. Ladik, J. Math. Phys. **16**, 598 (1975); **17**, 1011 (1976).
[2] R. Scharf and A. R. Bishop, Phys. Rev. A **43**, 6535 (1991).
[3] H.-H. Chen and C.-S. Liu, Phys. Rev. Lett. **37**, 693 (1976).
[4] R. Balakrishnan, Phys. Rev. A **32**, 1144 (1985).
[5] I. M. Besieris, in *Nonlinear Electromagnetics,* edited by P. L. E. Uslenghi (Academic, New York, 1980).
[6] V. V. Konotop, Phys. Rev. E **47**, 1473 (1993).
[7] See note added in Ref. [3].
[8] V. S. Gerdjikov, M. I. Ivanov, and P. P. Kulish, J. Math. Phys. **25**, 25 (1984).
[9] See, e.g., J. M. Ziman, *Principles of the Theory of Solids* (Cambridge University Press, Cambridge, 1964).
[10] In Ref. [2] the oscillating behavior of a soliton was treated using the concept of the effective potential. As it is clear, that description is in a good agreement with the approach proposed here.